Spectral Theory of the Linear-Quadratic and $\mathbf{H}^\infty$ Problems*

Edmond A. Jonckheere, Jyh-Ching Juang, and Leonard M. Silverman
Department of Electrical Engineering–Systems
University of Southern California
Los Angeles, California 90089-0781

Submitted by Paul A. Fuhrmann

ABSTRACT

An overview of the spectral theory of the linear-quadratic and $\mathbf{H}^\infty$ problems is presented. Originally, the spectral theory of the linear quadratic problem emerged out of the problem of characterizing the finite-time escape of the Riccati differential equation as a condition on the spectrum of a *Toeplitz-plus-Hankel* operator. Later and following an independent line of thought, the spectral theory of the $\mathbf{H}^\infty$ problem emerged as a Toeplitz-plus-Hankel operator characterization of the smallest achievable tolerance in feedback design. This common Toeplitz-plus-Hankel operator structure shared by the linear-quadratic and $\mathbf{H}^\infty$ problems is elucidated by mapping the usual $\mathbf{H}^\infty$ frequency-response specification into the time domain, leading to an inequality bound on a quadratic functional precisely induced by the Toeplitz-plus-Hankel operator. With this deep linear-quadratic–$\mathbf{H}^\infty$ connection at hand, we derive a linear-quadratic solution to the $\mathbf{H}^\infty$ problem, and show that such issues as the Adamjan-Arov-Krein problem, pole-zero cancellation in optimal $\mathbf{H}^\infty$ compensation, etc. can be fruitfully attacked using simple linear-quadratic arguments.

1. INTRODUCTION

The objective of this paper is to present a coherent exposition of a mathematical theory that fits within the intersection of the linear-quadratic and $\mathbf{H}^\infty$ problems. Historically, the fact that the linear-quadratic and $\mathbf{H}^\infty$ problems share some common mathematical structure was probably per-

*This research was supported by NSF grant DMS-8703954 and AFOSR grant 85-0256.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

ceived for the first time by Jonckheere and Verma [8], who observed that both the linear-quadratic and $\mathbf{H}^\infty$ theories, although apparently irreconcilable, enjoy a common *Toeplitz-plus-Hankel* operator structure.

Chronologically, the Toeplitz-plus-Hankel operator structure was first introduced by Jonckheere and Silverman [5–7] in the linear-quadratic context. To be a little more specific, consider the state-space system

$$\dot{x} = Ax + Bu$$

together with the performance index $x^TQx + 2x^TSu + u^TRu$ evaluated over the entire past history $(-\infty, 0]$. The resulting infinite-horizon, reverse-time cost is obviously a quadratic functional in $u \in \mathbf{L}^2(-\infty, 0]$; however, less obvious is the fact that the self-adjoint operator that induces this quadratic form is a Toeplitz operator $\mathbf{T}$ perturbed by a Hankel structure $\mathbf{H}_1^*\mathbf{H}_2$,

$$\int_{-\infty}^{0} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} d\tau = (u, (\mathbf{T} + \mathbf{H}_1^*\mathbf{H}_2)u).$$

As argued by Jonckheere and Silverman [5], this operator $\mathbf{T} + \mathbf{H}_1^*\mathbf{H}_2$, more precisely its spectrum, plays a crucial role in the linear-quadratic problem.

On the other hand, several years later and following an independent development, the $\mathbf{H}^\infty$ theory also revealed some Toeplitz-plus-Hankel structure. To grasp this structure in a simple setting, we consider a problem—the so-called two-block problem—that is central in the mixed sensitivity $\mathbf{H}^\infty$ design:

$$\inf_{Q \in \mathbf{H}_\perp^\infty} \left\| \begin{bmatrix} H - Q \\ V \end{bmatrix} \right\|_\infty = \epsilon_0,$$

where $H, V \in \mathbf{H}^\infty$. (Observe that we have departed from the usual conventions by interchanging the role of $\mathbf{H}^\infty$ and $\mathbf{H}_\perp^\infty$,[1] for reasons that will become clear later.) In the above, $Q \in \mathbf{H}_\perp^\infty$ denotes an arbitrary stabilizing compensator, H and V are completely specified by the plant and the frequency weighting, and the infimum ϵ_0 is the tightest tolerance that can be achieved by means of a feedback.[2] The remarkable fact is that this smallest achievable tolerance can be characterized independently of the compensator [8]: to be

[1] $\mathbf{H}_\perp^\infty$ stands for the set of rational functions with their poles in the open right half plane.
[2] We assume that $\epsilon_0 > \|V\|_\infty$.

more precise,

$$\epsilon_0^2 = \|\mathbf{T}_{V^*V} + \mathbf{H}_H^*\mathbf{H}_H\|.$$

The above Toeplitz-plus-Hankel characterization of the smallest achievable tolerance has been a milestone in the historical development of the $\mathbf{H}^{\infty}$ theory. The proof of the above, one way or the other, requires Nehari's theorem [8]. However, in this paper we shall show that Nehari's theorem can be discarded and we shall rather construct a self-contained proof that relies entirely on the linear-quadratic machinery.

At this juncture, the Toeplitz-plus-Hankel operator appears to be a common mathematical structure shared by the linear-quadratic and $\mathbf{H}^{\infty}$ problems. Going one step further, it was proved in [4, 8] that if we are given an $\mathbf{H}^{\infty}$ problem, then there exists a linear-quadratic problem such that

$$(\mathbf{T} + \mathbf{H}_1^*\mathbf{H}_2)_{\mathrm{LQ}} = (\mathbf{T}_{V^*V} + \mathbf{H}_H^*\mathbf{H}_H)_{\mathbf{H}^{\infty}}.$$

In view of this identification, it is possible to evaluate, in an efficient manner, the $\mathbf{H}^{\infty}$ tolerance ϵ_0 using the infinite-horizon reverse-time cost functional, itself related to the antistabilizing solution of the algebraic Riccati equation. This was, chronologically, the first motivation for establishing this $\mathbf{H}^{\infty}$-LQ link; see Jonckheere and Juang [4] and Jonckheere and Verma [8].

However, this common Toeplitz-plus-Hankel structure is just the tip of the iceberg, and the purpose of this paper is precisely to provide an in-depth exploration of the common mathematical structure shared by the linear quadratic and $\mathbf{H}^{\infty}$ problems. Conceptually, no results from the $\mathbf{H}^{\infty}$ theory are taken for granted. We rather proceed from the linear-quadratic theory, define a time-to-frequency-domain mapping that links the $\mathbf{H}^{\infty}$ and linear-quadratic problems in a conceptually clear way, and finally rebuild the $\mathbf{H}^{\infty}$ theory entirely from linear-quadratic arguments.

The present paper was inspired by, and is actually the written version of, the talk given by the first author at the Stanford Workshop organized by T. Kailath and S. Boyd in September 1987.

2. SPECTRAL THEORY OF THE LINEAR-QUADRATIC PROBLEM

In this section, we review the spectral theory of the linear-quadratic problem. Rather than presenting a comprehensive exposition—for which the reader is referred to Jonckheere and Silverman [5–7]—here we proceed from the classical issue of the finite-time escape of the Riccati equation and then develop the (less classical) Toeplitz-plus-Hankel operator characterization of this phenomenon. The spectral structure of the Toeplitz-plus-Hankel operator is then investigated, and finally this spectral structure is related to the antistabilizing solution P_- of the algebraic Riccati equation.

Consider the standard full-state-feedback linear-quadratic problem:

$$\dot{x} = Ax + Bu, \qquad x(i) = x_i,$$

$$J[i,t](x_i, u) = \int_i^t [x^T \quad u^T] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau.$$

In the above, (A, B) is controllable. It is further assumed that A is asymptotically stable. By feedback invariance, this incurs no loss of generality. The weighting matrix $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is symmetric, but it not necessarily positive semidefinite.[3] Closely associated with the linear-quadratic problem is the Riccati differential equation

$$-\dot{P} = A^T P + PA + Q - (PB + S)R^{-1}(PB + S)^T,$$

$$P(t) = 0.$$

Probably the easiest motivation for the spectral theory of the linear-quadratic problem is the issue of the *finite-time escape* of the Riccati differential equation. Assume that the overall weighting matrix $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is not positive semidefinite. Therefore as the initial time i recedes, there might occur a finite i at which the functional $J[i,t]$ ceases to be bounded from below for all $u \in \mathbf{L}^2[i,t]$, in which case the solution $P(\tau)$ of the Riccati differential equation has a finite-time escape at $\tau = i$.

To rephrase the above in a more formal setting, observe that J is a quadratic functional in both the initial condition x_i and the control u,

$$J[i,t] = x_i^T Q[i,t] x_i + 2x_i^T S[i,t] u + (u, R[i,t] u).$$

In the above, $Q[i,t]$ is a square, symmetric matrix and $S[i,t]$ is a bounded mapping: $\mathbf{L}^2[i,t] \to \mathscr{X}$. The computation of these quantities is left to the reader because they are not of crucial importance. Indeed, as far as boundedness of the functional $J[i,t]$ is concerned the most important mathematical object revealed by the above decomposition is the bounded, self-adjoint

[3] However, R is positive definite.

operator

$$R[i,t]:\mathbf{L}^2[i,t]\to\mathbf{L}^2[i,t]$$

with kernel

$$R[i,t](\alpha,\beta)=R\delta(\alpha-\beta)+\int_{\max(\alpha,\beta)}^{t}B^Te^{A^T(\tau-\alpha)}Qe^{A(\tau-\beta)}B\,d\tau$$

$$+\begin{cases}B^Te^{A^T(\beta-\alpha)}S & \text{for} \quad i\leqslant\alpha\leqslant\beta\leqslant t,\\ S^Te^{A(\alpha-\beta)}B & \text{for} \quad i\leqslant\beta\leqslant\alpha\leqslant t.\end{cases}$$

The fundamental role played by this operator is stated in the following, easily proved theorem (see [5,7]):

THEOREM 1. *Let the initial time i and the terminal time t be fixed. Then the following statements are equivalent.*

(a) *There exists a matrix* $M=M^T$ *such that* $J[i,t](x_i,u)\geqslant x_i^TMx_i$ *for all* x_i *and all* $u\in\mathbf{L}^2[i,t]$.

(b) $J[i,t](0,u)\geqslant 0\ \forall u\in\mathbf{L}^2[i,t]$.

(c) *The solution* $P(\tau)$ *of the Riccati differential equation exists, i.e., has no finite-time escape, over* $[i,t]$.

(d) $R[i,t]\geqslant 0$.

Now, consider the problem of guaranteeing no finite-time escape as the Riccati differential equation is integrated backward in time over an *arbitrarily large* interval, i.e., as $i\to-\infty$. From the foregoing theorem, this is equivalent to checking the positive semidefiniteness of $R[i,t]$ for all $i\leqslant t$. Since it involves an entire family of operators, this test is not very practical. The key to a more tractable test is the understanding of the nesting property of $R[i,t]$. To that effect, let the time instant j precede the time instant i:

$$j\leqslant i,$$

and consider the bottom square of the following ladder diagram:

$$\begin{array}{ccc}\mathbf{L}^2(-\infty,t] & \xrightarrow{R(-\infty,t]} & \mathbf{L}^2(-\infty,t]\\ \uparrow i & & \downarrow p\\ \mathbf{L}^2[j,t] & \xrightarrow{R[j,t]} & \mathbf{L}^2[j,t]\\ \uparrow i & & \downarrow p\\ \mathbf{L}^2[i,t] & \xrightarrow{R[i,t]} & \mathbf{L}^2[i,t]\end{array}$$

It is easily seen from the definition of $R[i,t]$ that the bottom square commutes; in other words, $R[i,t]$ is the restriction of $R[j,t]$ to $\mathbf{L}^2[i,t]$; therefore, $R[j,t] \geqslant 0$ guarantees $R[i,t] \geqslant 0$. More importantly, the positive-semidefiniteness test on the entire family of operators $\{R[i,t] : i \leqslant t\}$ could reduce to one single test on one single operator $R(-\infty,t]$ if such an operator —making the ladder diagram commute for all $j \leqslant i$—can be found. This operator exists and is easily seen to be

$$R(-\infty,t] : \mathbf{L}^2(-\infty,t] \to \mathbf{L}^2(-\infty,t]$$

with kernel

$$R(-\infty,t](\alpha,\beta) = R\delta(\alpha-\beta) + \int_{\max(\alpha,\beta)}^{t} B^T e^{A^T(\tau-\alpha)} Q e^{A(\tau-\beta)} B \, d\tau$$

$$+ \begin{cases} B^T e^{A^T(\beta-\alpha)} S & \text{for} \quad -\infty \leqslant \alpha \leqslant \beta \leqslant t, \\ S^T e^{A(\alpha-\beta)} B & \text{for} \quad -\infty \leqslant \beta \leqslant \alpha \leqslant t. \end{cases} \tag{1}$$

Therefore, one single positive semidefiniteness test on $R(-\infty,t]$ is sufficient to guarantee no finite-time escape, regardless of how far back in time the Riccati differential equation is integrated. To be more precise,

THEOREM 2. *The statements of Theorem* 1 *are verified for any* $i \leqslant t$ *if and only if* $R(-\infty,t] \geqslant 0$.

The above commutative-diagram argument has revealed an operator $R(-\infty,t]$ which is claimed to play a central role in the linear-quadratic problem. Indeed, the simple test $R(-\infty,t] \geqslant 0$ guarantees global existence of the solution of the Riccati differential equation as it is integrated backwards in time. More precisely, $R(-\infty,t] \geqslant 0$ is equivalent to demanding that $\operatorname{spec}(R(-\infty,t]) \subseteq \mathbf{R}^+$, so that the *spectrum* of $R(-\infty,t]$ appears to be of paramount importance. This motivates the terminology *spectral theory of the linear-quadratic problem.* In essence, the spectral theory of the linear-quadratic problem asserts that not only the finite-time escape but the whole structure of the linear-quadratic problem is lumped into the spectrum of $R(-\infty,t]$.

The operator $R(-\infty,t]$ has been defined as the operator that makes the diagram commute. However, at this stage, it would be interesting to relate this operator to some of the more conventional mathematical objects of the linear-quadratic theory. Define the functional

$$\mathbf{X}:\mathbf{L}^2(-\infty,t] \to \mathbf{L}^2(-\infty,t],$$

$$u \mapsto \mathbf{X}u(\tau) = \int_{-\infty}^{\tau} e^{A(\tau-\alpha)}Bu(\alpha)\,d\alpha.$$

In less precise but more intuitive language, $\mathbf{X}u(\cdot)$ is the solution to $\dot{x} = Ax + Bu$ subject to the initial condition $x(i = -\infty) = 0$. Then it is easily verified by direct calculation that the quadratic form induced by $R(-\infty, t]$ is the reverse-time, infinite-horizon functional

$$(u, R(-\infty,t]\,u) = \int_{-\infty}^{t} \begin{bmatrix} \mathbf{X}u^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \mathbf{X}u \\ u \end{bmatrix} d\tau.$$

The above can be considered as an alternative definition of $R(-\infty, t]$; it will be used extensively in the sequel.

At this juncture, any further interpretation of $R(-\infty, t]$ and the role it plays in the linear-quadratic theory requires knowledge of the structure of its spectrum. We proceed as in Jonckheere and Silverman [5] by first exploiting perturbation theory. Again, from (1) it can be verified by direct calculation that the kernel $R(-\infty, t](\alpha, \beta)$ can be split as follows:

$$R(-\infty,t](\alpha,\beta) = R\delta(\alpha-\beta) + \begin{cases} B^T e^{A^T(\beta-\alpha)}(S+YB), & -\infty < \alpha \leqslant \beta \leqslant t, \\ (B^TY + S^T)e^{A(\alpha-\beta)}B, & -\infty < \beta \leqslant \alpha \leqslant t \end{cases}$$

$$- B^T e^{A^T(t-\alpha)} Y e^{A(t-\beta)} B.$$

In the above $Y = Y^T$ is the solution of the Lyapunov equation

$$A^TY + YA = -Q.$$

More abstractly, the above decomposition reveals that $R(-\infty, t]$ is a perturbed Toeplitz operator

$$R(-\infty,t] = \mathbf{T}_\pi - \Phi^* Y \Phi. \tag{2}$$

Clearly Φ is the reachability map

$$\Phi : \mathbf{L}^2(-\infty, t] \to \mathscr{X},$$

$$u \mapsto \Phi u = \int_{-\infty}^{t} e^{A(t-\alpha)} B u(\alpha)\, d\alpha = x(t).$$

More importantly, the symbol π of the Toeplitz part is

$$\pi(j\omega) = R + B^T(-j\omega I - A^T)^{-1}(S + YB) + (B^T Y + S^T)(j\omega I - A)^{-1}B,$$

and some elementary manipulations reveal that π is exactly the celebrated Popov function

$$\pi(j\omega) = R + B^T(-j\omega I - A^T)^{-1}S + S^T(j\omega I - A)^{-1}B$$
$$+ B^T(-j\omega I - A^T)^{-1}Q(j\omega I - A)^{-1}B.$$

The fact that $R(-\infty, t]$ is a perturbation of $\mathbf{T}_\pi$ provides what is probably the most natural interpretation of the Popov function—the Popov function π of the linear quadratic problem is nothing other than the symbol of the Toeplitz part of $R(-\infty, t]$.

We now open a parenthesis to show how closely $R(-\infty, t]$ is related to the Toeplitz-plus-Hankel operator of the $\mathbf{H}^\infty$ problem. Factor the weighting matrix Q as follows:

$$Q = C_1^T C_2,$$

and define the observability map

$$\Omega_1 : \mathscr{X} \to \mathbf{L}^2[0, \infty),$$

$$x_0 \mapsto C_1 e^{A\tau} x_0.$$

Ω_2 is defined in a similar obvious way. Clearly,

$$Y = \Omega_1^* \Omega_2$$

and

$$R(-\infty, t] = \mathbf{T}_\pi - (\Omega_1 \Phi)^*(\Omega_2 \Phi).$$

The composition of the reachability and observability maps is clearly a Hankel operator, and therefore the above reveals the Toeplitz-plus-Hankel structure.

Putting together the above three paragraphs, we get

$$\int_{-\infty}^{0} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau = \left(u, \left[\mathbf{T}_\pi - (\Omega_1\Phi)^*(\Omega_2\Phi)\right] u\right),$$

which is apparently in flagrant contradiction with the well-known fact that

$$\int_{0}^{\infty} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau = (u, \mathbf{T}_\pi u) \qquad \text{for} \quad x(0) = 0.$$

The reason for the extra Hankel structure $(\Omega_1\Phi)^*(\Omega_2\Phi)$ in the $(-\infty, 0]$ case is that the control $u \in \mathbf{L}^2(-\infty, 0]$ generates a *terminal* state $x(0) = \Phi u$ which is itself responsible for the Hankel perturbation.

Now we follow classical perturbation theory [5–7] by first looking at the spectrum of $\mathbf{T}_\pi$ and then evaluating the extent to which the spectrum is perturbed by $\Phi^* Y \Phi$.

LEMMA 1.

$$\operatorname{ess\,spec}(\mathbf{T}_\pi) = \bigcup \{\lambda_i(\pi(jw)) : w \in \mathbf{R}\}',$$

where the prime denotes the closure.

LEMMA 2. *The spectrum of* $\mathbf{T}_\pi$ *is included in the convex hull of* $\operatorname{ess\,spec}(\mathbf{T}_\pi)$.

LEMMA 3. $\Phi^* Y \Phi$ *is compact.*

THEOREM 3.

$$\begin{aligned} \operatorname{ess\,spec}(R(-\infty, t]) &= \operatorname{ess\,spec}(\mathbf{T}_\pi) \\ &= \bigcup_i \{\lambda_i(\pi(jw)) : w \in \mathbf{R}\}'. \end{aligned}$$

More concretely, the above reveals that the spectrum of $R(-\infty, t]$ has an essential component $\bigcup_i \lambda_i(\pi(\text{II}))'$, i.e., the locus of the eigenvalues of π

evaluated on the imaginary axis II. Outside this essential spectrum, $R(-\infty, t]$ has, in general, some isolated eigenvalues of finite multiplicities. There may be finitely or infinitely many such eigenvalues. If there are infinitely many eigenvalues outside the essential spectrum, they accumulate on $\partial \bigcup_i \lambda_i(\pi(\text{II}))'$. It appears that perturbation theory is unable to rule out this pathological situation of infinitely many eigenvalues accumulating at the boundary of the essential spectrum. However, because of the finite-dimensional nature of the underlying linear-quadratic problem, intuition would rather tell us that there should be at most finitely many eigenvalues outside the essential spectrum. The proof of this fact, however, requires an argument far remote from perturbation theory. We outline a constructive proof [6] which yields a systematic procedure to compute all eigenvalues.

THEOREM 4. *There are at most finitely many eigenvalues with finite multiplicities in the spectrum of* $R(-\infty, t]$.

Proof. The key to the proof is a certain upper–lower-triangular factorization of $R(-\infty, t]$. There are many such factorizations, but one of them is particularly easy to get. Consider the following factorization of the weighting matrix:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C_1^T \\ D_1^T \end{bmatrix} \begin{bmatrix} C_2 & D_2 \end{bmatrix},$$

and define the transfer functions

$$G_i(s) = D_i + C_i(sI - A)^{-1}B, \qquad i = 1,2.$$

[Observe that this yields the following factorization of the Popov function: $\pi(s) = G_1^T(-s)G_2(s)$.] Consider the Toeplitz operators $\mathbf{T}_{G_1}$ and $\mathbf{T}_{G_2}$; these are actually lower triangular, Volterra operators. Therefore, it can be easily verified by straightforward computation that

$$R(-\infty, t] = \mathbf{T}_{G_1}^* \mathbf{T}_{G_2}.$$

Now, remember that the spectrum of an operator is invariant, modulo $\{0\}$, under commutation of the order of the factors:

$$\operatorname{spec}(R(-\infty, t]) - \{0\} = \operatorname{spec}(\mathbf{T}_{G_2}\mathbf{T}_{G_1}^*) - \{0\}$$

and observe that

$$\mathbf{T}_{G_2}\mathbf{T}_{G_1}^{*} = \mathbf{T}_{G_2 G_1^{*}}.$$

Therefore, up to the point $\{0\}$, the perturbed Toeplitz operator $R(-\infty, t]$ has the same spectrum as the Toeplitz operator $\mathbf{T}_{G_2 G_1^{*}}$ with *rational* symbol. As shown by Jonckheere and Silverman [6], given an arbitrary Toeplitz operator with rational symbol, we can construct a polynomial matrix, the zeros of which are discrete eigenvalues of the Toeplitz operator. As a corollary, $\mathbf{T}_{G_2 G_1^{*}}$ and hence $R(-\infty, t]$ have finitely many isolated eigenvalues with finite multiplicities. ■

To summarize, we have so far been able to prove the following

THEOREM 5. *The operator $R(-\infty, t]$ has an essential spectrum $\bigcup_i \{\lambda_i(\pi(jw)) : w \in \mathbf{R}\}'$. Outside the essential spectrum, there are at most finitely many isolated eigenvalues with finite multiplicities.*

REMARK. It has been conjectured ever since the original paper of Jonckheere and Silverman [5] that the number of finite eigenvalues outside the essential spectrum of $R(-\infty, t]$ never exceeds the dimension of the state. Lately, several proofs [8, 10], covering of broad class of linear-quadratic problems, have appeared. The general proof appears hard to construct. However, the issue of the number of eigenvalues is not essential as far as the ultimate objective of this paper is concerned.

REMARK. Since our concern is the spectrum of $R(-\infty, t]$, we have focused our attention on those finite eigenvalues located *outside* the essential spectrum, although finite eigenvalues embedded in the essential spectrum cannot be ruled out. However, for reasons explained in [5] we conjecture that this does not happen.

REMARK. Khargonekar [12] came close to proving the above conjectures. His argument is based on the utilization of transformations of the so-called *Riccati group* that act on the linear-quadratic data and yet leave the flow of the Riccati equation unchanged. In particular, there exists a transformation of the linear-quadratic data such that $R(-\infty, t]$ reduces to the identity plus a perturbation of rank not exceeding the dimension of the state. However, this result cannot be used in the $\mathbf{H}^{\infty}$ context, because Khargonekar's transformation does change the norm of $R(-\infty, t]$.

The spectral structure of $R(-\infty, t]$ reveals the limitation of the celebrated Popov test $\pi(jw) \geqslant 0\ \forall w \in \mathbf{R}$. It now appears that this is a test on the essential part of the spectrum of $R(-\infty, t]$ that gives no indication as to where the additional eigenvalues might be.

The rest of this section is devoted to linking the condition $R(-\infty, t] \geqslant 0$ to the algebraic Riccati equation. Looking at the reverse-time quadratic functional, define the problem

$$\hat{J}(x_t) = \inf_{\substack{u \in \mathbf{L}^2(-\infty, t] \\ x(t) = x_t}} \int_{-\infty}^{t} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau. \tag{3}$$

The infimum is to be taken over all controls $u \in \mathbf{L}^2(-\infty, t]$ driving the state into the preassigned terminal state x_t. In view of (2), this is the same as

$$\hat{J}(x_t) = \inf_{\substack{u \in \mathbf{L}^2(-\infty, t] \\ x(t) = x_t}} (u, \mathbf{T}_\pi u) - x_t^T Y x_t.$$

THEOREM 6. *The infimum exists for all x_t if and only if $\mathbf{T}_\pi \geqslant 0$; equivalently, if and only if $\pi(jw) \geqslant 0\ \forall w \in \mathbf{R}$. Furthermore, under the more restrictive situation $\mathbf{T}_\pi > 0$ or $\pi(jw) > 0\ \forall w$, the infimum is*

$$\hat{J}(x_t) = -x_t^T P_- x_t$$

with

$$P_- = Y - \left(\Phi \mathbf{T}_\pi^{-1} \Phi^*\right)^{-1}. \tag{4}$$

Proof. This follows from an easy Hilbert-space variational argument; see Jonckheere and Silverman [5]. Observe that since $\mathbf{T}_\pi^{-1} > 0$ and the reachability map Φ is onto, $\Phi \mathbf{T}_\pi^{-1} \Phi^*$ is invertible. ∎

Now, we link the above results to a solution of the algebraic Riccati equation:

$$A^T P + PA + Q - (PB + S) R^{-1} (PB + S)^T = 0. \tag{5}$$

THEOREM 7. *Assume $\pi(jw) \geqslant 0\ \forall w \in \mathbf{R}$. Then P_- as defined by* (4) *is the unique antistabilizing solution of the algebraic Riccati equation.*

Proof. It follows from a classical completion-of-the-squares argument. ∎

The following can be thought of as the Riccati-equation test for $R(-\infty, t] \geqslant 0$.

THEOREM 8. *Assume $\pi(jw) \geqslant 0$ $\forall w \in \mathbf{R}$. Then $R(-\infty, t] \geqslant 0$ if and only if $P_- \leqslant 0$.*

Proof. Easy from the variational definition of P_-. ∎

To complete this section, we look at an issue that will prove of paramount importance in the $\mathbf{H}^\infty$ problem. We assume that $\mathbf{T}_\pi > 0$, but that $R(-\infty, t]$ is not necessarily positive semidefinite. More concretely, we consider a situation characterized by an essential spectrum included in $\mathbf{R}^+$ and some eigenvalues around the origin. This typically occurs in a linear-quadratic problem characterized by $Q \geqslant 0$. In this situation, P_- exists but is not necessarily negative semidefinite. Our objective is to relate the inertia of P_- to the number of negative eigenvalues of $R(-\infty, t]$.

THEOREM 9. *Let $\mathbf{T}_\pi > 0$ and let $R(-\infty, t]$ be invertible. Then P_- is nonsingular and*

$$P_- = -\left\{\Phi\left(R(-\infty, t]\right)^{-1}\Phi^*\right\}^{-1}.$$

Proof. To prove the above, it suffices to verify the following matrix equality:

$$-\Phi\left(R(-\infty, t]\right)^{-1}\Phi^*\left[Y - \left(\Phi\mathbf{T}_\pi^{-1}\Phi^*\right)^{-1}\right] = I.$$

Elementary manipulation shows that the above is verified if and only if

$$-\Phi(\mathbf{T}_\pi - \Phi^* Y \Phi)^{-1}\Phi^*\left(Y\Phi\mathbf{T}_\pi^{-1}\Phi^* - I\right) = \Phi\mathbf{T}_\pi^{-1}\Phi^*,$$

or equivalently

$$-\Phi(\mathbf{T}_\pi - \Phi^* Y \Phi)^{-1}\left(\Phi^* Y\Phi\mathbf{T}_\pi^{-1} - I\right)\Phi^* = \Phi\mathbf{T}_\pi^{-1}\Phi^*.$$

To check the above it suffices to verify that

$$-(\mathbf{T}_\pi - \Phi^* Y \Phi)^{-1}\left(\Phi^* Y\Phi\mathbf{T}_\pi^{-1} - I\right) = \mathbf{T}_\pi^{-1},$$

and the latter is trivial. ∎

THEOREM 10. *Let $\mathbf{T}_\pi > 0$ and $R(-\infty, t] \geqslant 0$. Then $P_- < 0$ if and only if $R(-\infty, t]$ has no eigenvalues at zero. Furthermore, if $R(-\infty, t]$ has an eigenvalue with multiplicity μ at 0, then the kernel of P_- has dimension μ.*

Proof. That $P_- < 0$ iff $R(-\infty, t] > 0$ is trivial from the variational definition of P_-. To prove the other claim, let $v_1, v_2, \dots, v_\mu$ be μ linearly independent eigenvectors of $R(-\infty, t]$ associated with the eigenvalue at 0. It is claimed that these eigenvectors generate μ linearly independent terminal states $x_{t_i} = \Phi v_i$. Indeed, if they don't, then there exists a linear combination such that $\sum_i \alpha_i \Phi v_i = 0$. This together with $R(-\infty, t](\sum_i \alpha_i v_i) = 0$ yields $\mathbf{T}_\pi(\sum_i \alpha_i v_i) = 0$, which contradicts $\mathbf{T}_\pi > 0$. Therefore, $x_{t_i} = \Phi v_i$ constitute μ linearly independent terminal states induced by the eigenvectors. It then follows from the variational definition of P_- that $x_{t_i}^T P_- x_{t_i} = 0$ $\forall i$. Hence $\dim \ker(P_-) \geqslant \mu$. To complete the proof, it suffices to show that every terminal state x_t linearly independent of x_{t_i} generates a nonvanishing cost. Any control that drives the plant into such a state x_t has a nonvanishing $\{v_i\}^\perp$ component and, by virtue of the spectral decomposition of $R(-\infty, t]$, generates a nonvanishing cost. ■

The following is the major inertia result.

THEOREM 11. *Assume $\mathbf{T}_\pi > 0$. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < 0 < \lambda_{k+1} \cdots$ be the eigenvalues of $R(-\infty, t]$, and let μ_k be the dimension of the eigenspace of the negative eigenvalues $\lambda_1, \dots, \lambda_k$. Then P_- is nonsingular and has μ_k positive eigenvalues and $n - \mu_k$ negative eigenvalues.*

Proof. Let $v_1, \dots, v_{\mu_k}$ be linearly independent eigenvectors of the negative eigenvalues of $R(-\infty, t]$. It is claimed that they generate μ_k linearly independent states $x_{t_i} = \Phi v_i$. Assume they don't. Hence there exists a linear combination such that $\sum_i \alpha_i \Phi v_i = 0$. Since these eigenvectors are associated with negative eigenvalue, it follows that

$$\left(\sum_i \alpha_i v_i, R(-\infty, t] \sum_i \alpha_i v_i\right) < 0.$$

On the other hand, using $R(-\infty, t] = \mathbf{T}_\pi - \Phi^* Y \Phi$ yields

$$\left(\sum_i \alpha_i v_i, R(-\infty, t]\left(\sum_i \alpha_i v_i\right)\right) = \left(\sum_i \alpha_i v_i, \mathbf{T}_\pi\left(\sum_i \alpha_i v_i\right)\right) > 0.$$

The above two inequalities clearly contradict each other. Hence $\{x_{t_i} = \Phi v_i\}$ is a linearly independent set. Therefore, from the variational definition of P_- it follows that $x_{t_i}^T P_- x_{t_i} > 0$ for μ_k independent x_{t_i}. Hence

$$(\#\text{ positive eigenvalues of } P_-) \geqslant \mu_k.$$

On the other hand, from $P_- = -\{\Phi(R(-\infty, t])^{-1}\Phi^*\}^{-1}$ it follows that

$$(\#\text{ positive eigenvalues of } P_-) \leqslant \mu_k.$$

Clearly, the number of positive eigenvalues of P_- is μ_k. ■

As a corollary of the above, we are now in a position to state a result regarding the number of eigenvalues of $R(-\infty, t]$.

COROLLARY 1. *Let $\mathbf{T}_\pi > 0$. Then the number (multiplicity counted) of (finite-multiplicity) eigenvalues of $R(-\infty, t]$ located below its essential spectrum never exceeds the dimension of the state.*

3. LQ-$\mathbf{H}^\infty$ MAPPING

This section is the heart of the paper, for it develops an intimate LQ-$\mathbf{H}^\infty$ connection. We proceed from the two-block $\mathbf{H}^\infty$ problem, pick a tolerance level ϵ, and ask whether ϵ can be achieved for some compensator $Q(s) \in \mathbf{H}^\infty_\perp$. The essential idea is to map the $\mathbf{H}^\infty$ frequency-response inequality into the time domain using Parseval-like arguments. This results in an inequality between quadratic functionals defined over $\mathbf{L}^2(-\infty, 0]$, which, by virtue of the spectral theory of the linear-quadratic problem, yields an inequality of the form $\mathbf{T} + \mathbf{H}_1^*\mathbf{H}_2 = \mathbf{T}_{V^*V} + \mathbf{H}_H^*\mathbf{H}_H \leqslant \epsilon^2 I$. In other words, ϵ is achievable only if ϵ^2 is greater than the spectral radius of the Toeplitz-plus-Hankel operator.

Assume the compensator

$$Q(s) \in \mathbf{H}^\infty_\perp$$

achieves the specification

$$\left\| \begin{bmatrix} H - Q \\ V \end{bmatrix} \right\|_\infty \leqslant \epsilon, \tag{6}$$

where the $\mathbf{H}^\infty$ data are

$$H(s) = D_H + C_H(sI - A_H)^{-1}B_H \in \mathbf{H}^\infty,$$

$$V(s) = D_V + C_V(sI - A_H)^{-1}B_H \in \mathbf{H}^\infty.$$

The $\mathbf{H}^\infty$ specification is clearly equivalent to the $\mathbf{L}^2(\text{II})$ specification:

$$\int_{-\infty}^{\infty} \|(H-Q)u\|^2\, dw + \int_{-\infty}^{\infty} \|Vu\|^2\, dw \leqslant \epsilon^2 \int_{-\infty}^{\infty} \|u\|^2\, dw \qquad \forall u \in \mathbf{L}^2(\text{II}).$$

Now, the problem is to eliminate $Q(s)$ so as to come up with a condition involving the data $H(s)$ and $V(s)$ only. It is easily seen that the above inequality is verified for all $u \in \mathbf{L}^2$ if and only if it is verified for all $u \in \mathbf{H}^2_\perp$.[4] Therefore, for all $u \in \mathbf{H}^2_\perp$, we have $Qu \in \mathbf{H}^2_\perp$ and this yields

$$\int_{-\infty}^{\infty} \|(Hu)_+\|^2\, dw \leqslant \int_{-\infty}^{\infty} \|(Hu)_+ + (Hu)_- - Qu\|^2\, dw$$

Combining the above two inequalities yields

$$\int_{-\infty}^{\infty} \|(Hu)_+\|^2\, dw + \int_{-\infty}^{\infty} \|Vu\|^2\, dw \leqslant \epsilon^2 \int_{-\infty}^{\infty} \|u\|^2\, dw \qquad \forall u \in \mathbf{H}^2_\perp. \quad (7)$$

The heart of the matter is to rewrite the above frequency-domain $\mathbf{L}^2(\text{II})$ inequality into a time-domain $\mathbf{L}^2(-\infty, 0]$ inequality. Clearly, by Parseval's theorem

$$\int_{-\infty}^{\infty} \|u\|^2\, dw = 2\pi \int_{-\infty}^{0} u^T(\tau) u(\tau)\, d\tau.$$

Now we look at the second term of the left-hand side of the inequality

$$\int_{-\infty}^{\infty} \|Vu\|^2\, dw = \int_{-\infty}^{\infty} u^* V^* V u\, dw$$

$$= \int_{-\infty}^{\infty} u^*(F + F^*) u\, dw,$$

[4] This is equivalent to stating that a Laurent operator defined over $\mathbf{L}^2(-\infty, \infty)$ is positive semidefinite if and only if the Toeplitz operator, i.e., its restriction to $\mathbf{L}^2(-\infty, 0]$, is positive semidefinite. For a proof, see [5].

where

$$F(s):\begin{cases}\dot{x} = A_H x + B_H u\\ y = \left(B_H^T Y_V + D_V^T C_V\right)x + \frac{1}{2}D_V^T D_V u\end{cases}$$

and Y_V is solution of the Lyapunov equation

$$A_H^T Y_V + Y_V A_H = -C_V^T C_V.$$

Hence,

$$\int_{-\infty}^{\infty} \|Vu\|^2\, dw = \int_{-\infty}^{\infty} y^T(-jw)u(jw)\, dw + \int_{-\infty}^{\infty} u^T(-jw)y(jw)\, dw$$

$$= 2\pi\int_{-\infty}^{0} y^T(\tau)u(\tau)\, d\tau + 2\pi\int_{-\infty}^{0} u^T(\tau)y(\tau)\, d\tau$$

$$= 2\pi\int_{-\infty}^{0}\left[x^T \quad u^T\right]\begin{bmatrix} 0 & Y_V B_H + C_V^T D_V\\ B_H^T Y_V + D_V^T C_V & D_V^T D_V\end{bmatrix}\begin{bmatrix} x\\ u\end{bmatrix} d\tau.$$

Finally, we look at the first term of the frequency-response inequality

$$\int_{-\infty}^{\infty} \|(Hu)_+\|^2\, dw = 2\pi\int_0^{\infty}\|C_H e^{A_H\tau}x_0\|^2\, d\tau$$

$$= 2\pi x_0^T Y_H x_0.$$

Clearly, Y_H is the solution of the Lyapunov equation

$$A_H^T Y_H + Y_H A_H = -C_H^T C_H, \tag{8}$$

and $x_0 = \Phi u$, where Φ is the reachability map. Hence

$$\int_{-\infty}^{\infty} \|(Hu)_+\|^2\, dw = 2\pi(u, \Phi^* Y_H \Phi u)_{L^2(-\infty,0]}.$$

From Section 2, we have

$$(u, \Phi^* Y_H \Phi u) = \int_{-\infty}^{0}\left(-\|C_H x\|^2\right) d\tau + \left(u, \mathbf{T}_{H_0^* H_0} u\right),$$

where

$$H_0(s) = C_H(sI - A_H)^{-1}B_H.$$

Observe that

$$H_0^* H_0 = B_H^T Y_H (j\omega I - A_H)^{-1} B_H + B_H^T(-j\omega I - A_H^T)^{-1} Y_H B_H.$$

Therefore,

$$(u, \Phi^* Y_H \Phi u) = \int_{-\infty}^{0} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} -C_H^T C_H & Y_H B_H \\ B_H^T Y_H & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau.$$

Finally, combining all of the above, the $\mathbf{L}^2$ frequency-response inequality yields

$$\int_{-\infty}^{0} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} -C_H^T C_H & (Y_H + Y_V)B_H + C_V^T D_V \\ D_V^T C_V + B_H^T(Y_H + Y_V) & D_V^T D_V \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau$$

$$\leqslant \epsilon^2 \int_{-\infty}^{0} u^T u \, d\tau. \quad (9)$$

To summarize the situation, the frequency-response $\mathbf{H}^\infty$ specification (6) is verified only if the time-domain inequality (9) is verified. This reverse-time quadratic-functional inequality can be rewritten in terms of the Toeplitz-plus-Hankel operator of the linear-quadratic problem. Elementary calculus is enough to compute this operator, and it yields

$$\mathbf{T}_{V^*V} + \mathbf{H}_H^* \mathbf{H}_H \leqslant \epsilon^2 I,$$

although this result was already obvious from (7).

Clearly the Toeplitz-plus-Hankel operator of a certain linear-quadratic problem has been equated to $\mathbf{T}_{V^*V} + \mathbf{H}_H^* \mathbf{H}_H$. To be more formal,

THEOREM 12 (LQ-$\mathbf{H}^\infty$ mapping). *Given an $\mathbf{H}^\infty$ problem*

$$\begin{bmatrix} H(s) \\ V(s) \end{bmatrix} = \begin{bmatrix} D_H \\ D_V \end{bmatrix} + \begin{bmatrix} C_H \\ C_V \end{bmatrix} (sI - A_H)^{-1} B_H,$$

there exists a linear-quadratic problem

$$A = A_H,$$

$$B = B_H,$$

$$Q = -C_H^T C_H,$$

$$S = (Y_H + Y_V)B_H + C_V^T D_V,$$

$$R = D_V^T D_V,$$

where

$$A_H^T(Y_H + Y_V) + (Y_H + Y_V)A_H^T = -C_H^T C_H - C_V^T C_V,$$

such that

$$\mathbf{T}_\pi - \Phi^* Y \Phi = \mathbf{T}_{V^*V} + \mathbf{H}_H^* \mathbf{H}_H.$$

This is clearly consistent with the data-matching recipe of Jonckheere and Juang [4]. However, in addition to rediscovering this recipe, the above frequency-to-time-domain transcription has provided the natural explanation of the deep connection between the $\mathbf{H}^\infty$ and linear-quadratic problems: Transcribing the $\mathbf{H}^\infty$ frequency-response tolerance specification into the time domain yields a reverse-time infinite-horizon quadratic-functional inequality.

Finally, it should be clear to the reader that in this section we have proved that the $\mathbf{H}^\infty$ tolerance ϵ is achievable *only if* $\mathbf{T}_{V^*V} + \mathbf{H}_H^* \mathbf{H}_H \leqslant \epsilon^2 I$. In the next section, we prove the *if* part, using a self-contained linear-quadratic argument, to be consistent with our objective of rederiving the $\mathbf{H}^\infty$ theory from the linear-quadratic theory.

4. LINEAR-QUADRATIC SOLUTION TO TWO-BLOCK $\mathbf{H}^\infty$ PROBLEM

In this section, *conceptually*, we proceed from the operator inequality $\mathbf{T}_{V^*V} + \mathbf{H}_H^* \mathbf{H}_H \leqslant \epsilon^2 I$, we determine the linear-quadratic implication of this inequality in terms of the algebraic Riccati equation, and with the solution of the algebraic Riccati equation, we construct a linear-quadratic state-space solution $Q(s) \in \mathbf{H}_\perp^\infty$ that achieves the $\mathbf{H}^\infty$ tolerance ϵ. We then repeat the same line of argument, starting from a tolerance ϵ^2 somewhere between two eigenvalues of $\mathbf{T}_{V*V} + \mathbf{H}_H^*\mathbf{H}_H$, and develop a linear-quadratic approach to the two-block Adamjan-Arov-Krein problem.

Consider the two-block $\mathbf{H}^\infty$ problem

$$\inf_{Q \in \mathbf{H}_\perp^\infty} \left\| \begin{bmatrix} H - Q \\ V \end{bmatrix} \right\|_\infty ,$$

where we assume, without loss of generality, that

$$H(s) = D_H + C_H(sI - A_H)^{-1}B_H \in \mathbf{H}^\infty,$$

$$V(s) = D_V + C_V(sI - A_H)^{-1}B_H \in \mathbf{H}^\infty,$$

i.e.,

$$\operatorname{Re} \lambda_i(A_H) < 0 \qquad \forall i.$$

The central idea is that one can associate with the $\mathbf{H}^\infty$ problem a (full-state-feedback) linear-quadratic problem:

$$\dot{x} = Ax + Bu,$$

$$\int_{-\infty}^{0} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau.$$

The driving motivation for establishing the link between the linear-quadratic and $\mathbf{H}^\infty$ problems is that their respective Toeplitz-plus-Hankel operators can be matched through the linear-quadratic–$\mathbf{H}^\infty$ mapping as in Theorem 12. Now, we come to the major result of this paper:

THEOREM 13. *Let us fix an arbitrary tolerance level* $\epsilon^2 > \|\mathbf{T}_{V*V} + \mathbf{H}_H^*\mathbf{H}_H\|$. *Then a compensator* $Q(s)$ *that achieves*

$$\left\| \begin{bmatrix} H - Q \\ V \end{bmatrix} \right\|_\infty \leqslant \epsilon$$

is given by

$$Q(s) = D_H + C_H(sP + A_H^T P + C_H^T C_H)^{-1}[(Y_H + Y_V)B_H + C_V^T D_V], \quad (10)$$

where $P < 0$ is the negative definite, antistabilizing solution of the algebraic Riccati equation

$$A^T P + PA - Q - (PB - S)(\epsilon^2 I - R)^{-1}(PB - S)^T = 0. \tag{11}$$

Before going through the proof of the theorem, a few remarks are in order.

REMARK. It is easily seen that the above algebraic Riccati equation is generated by the linear quadratic problem

$$\dot{x} = Ax + Bu,$$

$$\int_{-\infty}^{0} \epsilon^2 u^T u - \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau,$$

and clearly the Toeplitz-plus-Hankel operator of this problem is $\epsilon^2 I - (\mathbf{T}_{V^*V} + \mathbf{H}_H^* \mathbf{H}_H)$. If the tolerance ϵ is suboptimal, i.e., if $\epsilon^2 I - \mathbf{T}_{V^*V} - \mathbf{H}_H^* \mathbf{H}_H > 0$, then $\epsilon^2 I - \mathbf{T}_{V^*V} - \mathbf{H}_H^* \mathbf{H}_H$ has no eigenvalue at zero. Now, we invoke the spectral theory of the linear-quadratic problem, from which it follows that $\epsilon^2 I - \mathbf{T}_{V^*V} - \mathbf{H}_H^* \mathbf{H}_H \geqslant 0$ is enough to guarantee the *existence* of a negative semidefinite, antistabilizing solution P of the algebraic Riccati equation. We further invoke another result of the spectral theory: The fact that $\epsilon^2 I - \mathbf{T}_{V^*V} - \mathbf{H}_H^* \mathbf{H}_H$ has no eigenvalue at 0 guarantees $P < 0$. Therefore P is nonsingular and the degree of $Q(s)$ is at most the size n of A_H.

REMARK. Let us make sure that $Q(s)$ is indeed in $\mathbf{H}_\perp^\infty$. Since P is nonsingular, the algebraic Riccati equation can be manipulated to

$$\begin{aligned}(A_H^T + C_H^T C_H P^{-1})P + P(A_H + P^{-1} C_H^T C_H)& \\ = C_H^T C_H + (PB_H - S)(\epsilon^2 I - R)^{-1}(PB_H - S)^T.& \end{aligned} \tag{12}$$

Reinterpreting the above as the existence of a solution $P < 0$ to a Lyapunov equation implies that $A_H^T + C_H^T C_H P^{-1}$ is stable. Hence $Q(s) \in \mathbf{H}_\perp^\infty$.

REMARK. The theorem clearly provides an *all-linear-quadratic solution* to the two-block $\mathbf{H}^\infty$ problem. The essential idea is, "Given an $\mathbf{H}^\infty$ problem, define an adjoint linear quadratic problem, compute the solution to the Riccati equation, and with that Riccati solution, construct the solution Q of the $\mathbf{H}^{\infty}$ problem."

REMARK. The above theorem has, in fact, already been proved by Juang and Jonckheere [11]. However, the idea of that proof was to extend the one-block $\mathbf{H}^{\infty}$ solution of Glover [3] to the two-block $\mathbf{H}^{\infty}$ problem and then observe that the resulting solution could indeed be rewritten in terms of linear-quadratic data. This approach did not answer the question *why* the $\mathbf{H}^{\infty}$ problem admits a simple linear-quadratic solution. In the following, we provide a self-contained proof that involves only *simple* linear-quadratic arguments.

Proof. Define

$$S(s)=\begin{bmatrix} H(s)-Q(s) \\ V(s) \end{bmatrix},$$

and observe that the tolerance requirement can be rewritten as

$$\epsilon^2 I - S^T(-jw)S(jw) \geqslant 0.$$

The above is clearly the condition that $(1/\epsilon)S(s)$ be a bounded real scattering or Schur function, except for one point—$S(s)$ has both left- and right-half-plane poles.

LEMMA 4. *Consider* $S(s)=D_a+C_a(sI-A_a)^{-1}B_a$. *Then*

$$\epsilon^2 I - S^T(-jw)S(jw) \geqslant 0$$

if there exists a solution P_a *to the so-called bounded real algebraic Riccati equation*

$$A_a^T P_a + P_a A_a + C_a^T C_a + \left(P_a B_a + C_a^T D_a\right)\left(\epsilon^2 I - D_a^T D_a\right)^{-1}\left(P_a B_a + C_a^T D_a\right)^T = 0.$$

Proof of lemma. The algebraic Riccati equation implies the existence of a solution $(-P_a)$ to the linear matrix inequality

$$\begin{bmatrix} A_a^T(-P_a)+(-P_a)A_a - C_a^T C_a & (-P_a)B_a - C_a^T D_a \\ B_a^T(-P_a) - D_a^T C_a & \epsilon^2 I - D_a^T D_a \end{bmatrix} \geqslant 0. \tag{13}$$

Postmultiplying by

$$\begin{bmatrix} (jwI - A_a)^{-1}B_a \\ I \end{bmatrix}$$

and premultiplying by the complex conjugate transpose yields $\epsilon^2 I - S^T(-jw)S(jw) \geqslant 0$. ■

Proof (continued). It is easily seen that the state-space representation for $\begin{bmatrix} H-Q \\ V \end{bmatrix}$ is

$$A_a = \begin{bmatrix} A_H & 0 \\ 0 & -A_H^T - C_H^T C_H P^{-1} \end{bmatrix},$$

$$B_a = \begin{bmatrix} B_H \\ (Y_H + Y_V)B_H + C_V^T D_V \end{bmatrix},$$

$$C_a = \begin{bmatrix} C_H & -C_H P^{-1} \\ C_V & 0 \end{bmatrix},$$

$$D_a = \begin{bmatrix} 0 \\ D_V \end{bmatrix}.$$

Now, it remains to prove the existence of a solution to the bounded real Riccati equation. It is easily seen that this equation is verified for

$$P_a = \begin{bmatrix} Y_H + Y_V & -I \\ -I & P^{-1} \end{bmatrix}.$$ ■

The above proof has revealed a very general procedure to tackle $\mathbf{H}^\infty$ problems—the $\mathbf{H}^\infty$ tolerance requirement amounts to demanding that the transfer function $S(s)$ be bounded real. To exploit this characterization, combine the state-space equations of the plant and the compensator into 2×2-block-partitioned state space data (A_a, B_a, C_a, D_a). Finally, compute the 2×2-block-partitioned solution P_a, if any, of the bounded real Riccati equation. The 2×2 bounded real Riccati equation reduces to anything between two Lyapunov equations and two Riccati equations, depending on the complexity of the problem. In the above proof of the two-block case, the bounded real Riccati equation reduces to one Lyapunov equation for $Y_H + Y_V$ and one Riccati equation for P.

The linear-quadratic machinery that has been developed so far allows us to visualize the difficulty, already perceived in [5], that arises as ϵ decreases to the optimal tolerance level ϵ_0. Indeed, from the spectral theory of the linear-quadratic problem, it follows that at optimality P is negative semidefinite with a μ-dimensional kernel, where μ is the multiplicity of the largest eigenvalue of $\mathbf{T}_{V*V} + \mathbf{H}_H^* \mathbf{H}_H$. Therefore, it appears that the optimal tolerance can be reached, but with a descriptor $Q(s)$ that has generically μ dynamic modes at infinity. The achievement of the optimal tolerance with a well-behaved $Q(s)$, however, requires some tedious manipulations that are relegated to the Appendix.

At this stage, we have recovered, in a self-contained manner, the following key result:

COROLLARY 2. *The optimum tolerance level* ϵ_0^2 *is the spectral radius of* $\mathbf{T}_{V*V} + \mathbf{H}_H^* \mathbf{H}_H$.

Finally, we look at the Adamjan-Arov-Krein problem of assessing the level of tolerance that can be achieved with a $Q(s)$ that is allowed to have a limited number of stable poles in addition to an essentially unlimited number of unstable poles. By the same token, this provides an interpretation of *all* eigenvalues of $T_{V*V} + \mathbf{H}_H^* \mathbf{H}_H$.

THEOREM 14. *Let* $\lambda_1 > \lambda_2 > \cdots$ *be the eigenvalues of* $\mathbf{T}_{V*V} + \mathbf{H}_H^* \mathbf{H}_H$ *located above the essential spectrum. Pick a tolerance*

$$\lambda_k < \epsilon^2 < \lambda_{k-1},$$

and let μ *be the total multiplicity of* $\lambda_{k-1}, \dots, \lambda_1$. *Then the tolerance* ϵ *can be achieved with a* $Q(s)$ *that has at most* μ *stable poles. This* $Q(s)$ *is still given by* (10), *and* P *is the non-sign-definite, antistabilizing solution of the algebraic Riccati equation* (11).

Proof. The linear-quadratic problem that generates the algebraic Riccati equation is characterized by $\epsilon^2 I - \mathbf{T}_{V*V} - \mathbf{H}_H^* \mathbf{H}_H$. By hypothesis, the continuous spectrum is included in $\mathbf{R}^+$; which guarantees the existence of P. Since $\epsilon^2 I - \mathbf{T}_{V*V} - \mathbf{H}_H^* \mathbf{H}_H$ is invertible with μ negative eigenvalues, it follows that P is nonsingular and has μ positive and $n - \mu$ negative eigenvalues; see Theorem 11. From the inertia of the Lyapunov equation (12), it follows that $A_H^T + C_H^T C_H P^{-1}$ has $n - \mu$ stable and μ unstable eigenvalues. Therefore $Q(s)$

has at most μ stable poles. The proof that $Q(s)$ does achieve the tolerance ϵ is the same as that of Theorem 13. ■

5. CONCLUDING REMARKS

Lately, several state-space solutions to the $\mathbf{H}^\infty$ problem—resembling up to a certain extent our result of Theorem 13—have appeared [1,2]. Incidentally, in [11] we have proved that our solution of Theorem 13 after a change of Riccati solution variable yields the Doyle-Glover solution [2] in the two-block case.

That is to say that the *major* contribution of the present paper is rather *conceptual*. Essentially, we have shown that an $\mathbf{H}^\infty$ problem can be mapped to a (full-state-feedback) linear-quadratic problem, and vice versa. This mapping is nothing other than a Parseval-like frequency-domain-to-time-domain mapping. The $\mathbf{H}^\infty$ tolerance requirement is translated into a quadratic-functional inequality, itself equivalent to a Toeplitz-plus-Hankel operator inequality, the latter being the essence of the spectral theory of the linear-quadratic problem developed by Jonckheere and Silverman [5–7]. Consequently, state-space solutions to the $\mathbf{H}^\infty$ problem along with the Adamjan-Arov-Krein theory have been derived using simple, self-contained linear-quadratic arguments. This is believed to be of great didactic value.

APPENDIX. THE OPTIMAL CASE

The characterization of all interpolants of the two-block problem can be routinely derived using the results of, for example, Glover [3]; see Juang [9]. Here, in order to preserve the linear-quadratic nature of the argument and simplify the discussion, we make the following assumptions:

(a) $\epsilon_0 = 1$.
(b) $\|R\| < 1$.
(c) $H(s)$ is square.
(d) An interpolant $Q(s)$ of degree $n - \mu$ is considered.
(e) The all-pass solution is considered.

The above assumptions can be relaxed with slight modification. It is known that at optimality, the matrix P is singular (or rank $n - \mu$). Assume that P admits the following decomposition:

$$P = U \Sigma U^T, \tag{14}$$

where U is a matrix of dimension $n \times (n-\mu)$ with mutually orthogonal columns and Σ is a $(n-\mu)\times(n-\mu)$ (strictly) negative definite matrix. Injecting (14) in the algebraic Riccati equation (11) and further premultiplying and postmultiplying by the projection $(UU^T - I)$ yields an orthogonal matrix J such that

$$(UU^T - I)C_H^T J = -(UU^T - I)S(I-R)^{-1/2},$$

where $(I-R)^{1/2}$ is the symmetric square root of $I - R$.

PROPOSITION 1. *Under the above assumptions, the antistable transfer function*

$$Q(s) = D_Q + C_Q(sI - A_Q)^{-1}B_Q,$$

where

$$A_Q = -U^T A^T U + U^T Q U \Sigma^{-1} - U^T C_H^T J(I-R)^{-1/2}[-B^T U + SU\Sigma^{-1}],$$

$$B_Q = U^T S + U^T C_H^T J(I-R)^{1/2},$$

$$C_Q = C_H U \Sigma^{-1} + J(I-R)^{-1/2}[-B^T U + SU\Sigma^{-1}],$$

$$D_Q = D_H - J(I-R)^{1/2},$$

satisfies

$$\left\| \begin{bmatrix} H - Q \\ V \end{bmatrix} \right\|_\infty = 1.$$

Proof. Consider the following state-space realization of $\begin{bmatrix} H(s) - Q(s) \\ V(s) \end{bmatrix}$:

$$A_a = \begin{bmatrix} A_H & 0 \\ 0 & A_Q \end{bmatrix}, \qquad B_a = \begin{bmatrix} B_H \\ B_Q \end{bmatrix},$$

$$C_a = \begin{bmatrix} C_H & -C_Q \\ C_V & 0 \end{bmatrix}, \qquad D_a = \begin{bmatrix} D_H - D_Q \\ D_V \end{bmatrix}.$$

It is easily seen by direct calculation that

$$P_a = \begin{bmatrix} Y_H + Y_V & -U \\ -U^T & \Sigma^{-1} \end{bmatrix}$$

verifies the linear matrix inequality (13) of the bounded real lemma. Hence $Q(s)$ achieves the tolerance. To prove that $Q(s)$ is antistable, injecting (14) in (11), premultiplying by U^T, and postmultiplying by U yields

$$-\Sigma^{-1}A_Q - A_Q^T\Sigma^{-1} = C_Q^T C_Q,$$

and the antistability of $Q(s)$ follows from a classical Lyapunov argument. ■

The authors are grateful to Professor F. Callier, University of Namur, Belgium, for many constructive comments.

REFERENCES

1 J. Ball and N. Cohen, Sensitivity minimization in $\mathbf{H}^\infty$ norm: Parameterization of all suboptimal solutions, *Internat. J. Control* 46:685–716 (1987).

2 J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, State space solution to standard $\mathbf{H}^2$ and $\mathbf{H}^\infty$ control problems, in *Proceedings of American Control Conference*, 1988.

3 K. Glover, All optimal Hankel-norm approximation of linear multivariable system and their $\mathbf{L}^\infty$-error bounds, *Internat. J. Control* 39(6):1115–1193 (June 1984).

4 E. A. Jonckheere and J. C. Juang, Fast computation of achievable performance in mixed sensitivity $\mathbf{H}^\infty$ design, *IEEE Trans. Automat. Control* AC-32(10):896–906 (1987).

5 E. A. Jonckheere and L. M. Silverman, Spectral theory of the linear quadratic optimal control problem: Discrete time single input case, *IEEE Trans. Circuits and Systems* CAS-25:810–825 (1978).

6 E. A. Jonckheere and L. M. Silverman, Spectral theory of the linear quadratic optimal control problem: A new algorithm for spectral computations, *IEEE Trans. Automat. Control* AC-25:880–888 (1980).

7 E. A. Jonckheere and L. M. Silverman, The linear quadratic optimal control problem—the operator theoretic viewpoint, *Oper. Theory Adv. Appl.* 12:277–302 (1984).

8 E. A. Jonckheere and M. Verma, A spectral characterization of $\mathbf{H}^\infty$ optimal feedback performance and its efficient computation, *Systems Control Lett.* 8:13–22 (1986).

9 J. C. Juang, $\mathbf{H}^\infty$ Design: Its Structure and Computations, Ph.D. Thesis, Univ. of Southern California, 1987.

10 J. C. Juang and E. A. Jonckheere, On computing the spectral radius of the Hankel plus Toeplitz operator, *IEEE Trans. Automat. Control* AC-23, Nov. 1988.

11 J. C. Juang and E. A. Jonckheere, Data reduction in the mixed-sensitivity $\mathbf{H}^{\infty}$ design problem, in *Proceedings of IEEE Conference on Decision and Control*, Los Angeles, 1987, pp. 1627–1632; see also *IEEE Trans. Automat. Control* AC-24, June 1989.

12 P. Khargonekar, Canonical Forms for Linear Quadratic Optimal Control Problems, Ph.D. Thesis, Univ. of Florida, 1981.

Received 19 September 1988; final manuscript accepted 8 November 1988